# A PROOF OF KAROUBI'S THEOREM $W(A[t]) = W(A)$

## 1. Introduction

Let $A$ be an associative ring with an involution denoted by $a \mapsto \overline{a}$. If $M$ is a right $A$-module we denote by $M^*$ its dual $\mathrm{Hom}(M, A)$ endowed with the right action of $A$ given by $fa(x) = \overline{a}f(x)$ for any $f : M \to A$ and $a \in A$. If $P$ is a finitely generated projective right $A$-module we identify it with $P^{**}$ through the canonical isomorphism mapping $x \in P$ to $\widehat{x} : P^* \to A$ defined by $\widehat{x}(f) = f(x)$.

Assume now that 2 is invertible in $A$ and let $\epsilon$ be 1 or $-1$. An *$\epsilon$-hermitian space* over $A$ is a pair $(P, f)$ consisiting of a finitely generated projective right $A$-module $P$ and an $A$-isomorphism $f : P \to P^*$ satisfying $f = \epsilon f^*$. For brevity $\epsilon$-hermitian spaces will be called *spaces.* A 1-hermitian space (over a commutative ring $A$) is also called *quadratic space.*

Two spaces $(P, f)$ and $(Q, g)$ are *isometric* if there exists an $A$-isomorphism $\varphi : P \to Q$ such that the square

$$\begin{tikzcd} P \arrow[r, "\varphi"] \arrow[d, "f"'] & Q \arrow[d, "g"] \\ P^* & Q^* \arrow[l, "\varphi^*"] \end{tikzcd}$$

commutes. A space is *hyperbolic* if it is isometric to a space of the form

$$H(P) = \left( P \oplus P^*, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) .$$

The *orthogonal sum* of two spaces $(P, f)$ and $(Q, g)$ is the space

$$(P, f) \perp (Q, g) = (P \oplus Q, f \oplus g) .$$

If $(P, f)$ is a space and $M$ a submodule of $P$ we denote by $M^\perp$ the orthogonal of $M$, defined by the exact sequence

$$0 \longrightarrow M^\perp \longrightarrow P \xrightarrow{i^* f} M^* ,$$

where $i^*$ is the dual of the inclusion $i : M \to P$. A submodule $M$ of $P$ is *totally isotropic* if $M \subseteq M^\perp$.

The Witt group $W(A)$ of $\epsilon$-hermitian spaces over $A$ is the quotient of the Grothendieck group of $\epsilon$-hermitian spaces with respect to orthogonal sums, by the subgroup generated by hyperbolic spaces. We say that two spaces are Witt equivalent if they represent the same element of $W(A)$. We recall three elementary well-known facts about hermitian spaces.

**Proposition 1.** *Let $(P, f)$ be any space. Then*

*1. The space $(P, f) \perp (P, -f)$ is hyperbolic.*

*2. If $M$ is a direct summand of $P$ and $M = M^\perp$, then $P$ is isometric to $H(M)$.*

*3. If $M$ is a totally isotropic direct summand of $P$, the map $f$ induces on $M^\perp/M$ a natural structure of hermitian space that makes it Witt equivalent to $(P, f)$.*

## 2. K-theoretic preliminaries

We recall a few results proved in the twelfth chapter of [1]. For any ring $A$ we denote by $K_0(A)$ the Grothendieck group of finitely generated projective right $A$-modules and by $K_1(A)$ the abelianized general linear group of $A$: $K_1(A) = GL(A)/[GL(A), GL(A)]$. By Whitehead's lemma $K_1(A)$ is also the quotient of $GL(A)$ by the subgroup $E(A)$ generated by all elementary matrices over $A$.

For any functor $F$ from rings to abelian groups we denote by $N_+F(A)$ the kernel of the map $F(A[t]) \to F(A)$ obtained by putting $t = 0$. Similarly, we denote by $N_-F(A)$ the kernel of $F(A[t^{-1}]) \to F(A)$ obtained by putting $t^{-1} = 0$. The inclusions of $A[t]$ and $A[t^{-1}]$ into $A[t, t^{-1}]$ define a map

$$N_+F(A) \oplus N_-F(A) \longrightarrow F(A[t, t^{-1}])$$

whose cokernel will be denoted by $LF(A)$. The functor $LK_1$ turns out to be naturally isomorphic to $K_0$, hence we will denote $LK_i$ by $K_{i-1}$ for $i = 1$ and also for $i = 0$.

**Proposition 2.** *Every element of $N_+(A)$ can be represented by a matrix of the form $1 + \nu t$, where $\nu$ is a nilpotent matrix in $GL(A)$.*

*Proof.* Let $\alpha = \alpha_0 + \alpha_1 t + \cdots + \alpha_m t^m$ be a matrix in $GL(A[t])$, with constant matrices $\alpha_0, \ldots, \alpha_m$. Assume that $m \geq 2$. The matrix

$$\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$$

is equivalent to $\alpha$ and, multiplying it on the right with an elementary matrix of degree 1, we can transform it into

$$\begin{pmatrix} \alpha & t \\ 0 & 1 \end{pmatrix} .$$

Multiplication on the left with an elementary matrix of degree $m - 1$ yields a matrix of degree $m - 1$ in the same class as $\alpha$. Repeating this procedure $m - 1$ times yields a representative of $\alpha$ of degree 1 in $t$. In our case $\alpha_0 = \alpha(0)$ is elementary, hence, multiplying $\alpha$ by $\alpha_0^{-1}$ we may assume that $\alpha_0 = 1$. It is easy to check that a matrix of the form $1 + \nu t$ is invertible if and only if $\nu$ is nilpotent.

**Theorem 3.** *Let $A$ be any associative ring.*
*(a) For $i = 0, 1$ there exists a natural embedding*

$$\lambda_i : K_{i-1}(A) \longrightarrow K_i(A[t, t^{-1}])$$

*such that the composite*

$$K_{i-1}(A) \xrightarrow{\lambda_i} K_i(A[t, t^{-1}]) \longrightarrow LK_i(A) = K_{i-1}(A)$$

*is the identity.*
*(b) The embeddings $\lambda_i$ and the canonical homomorphisms $N_\pm K_i(A) \to K_i(A[t, t^{-1}])$ yield canonical decompositions*

$$K_1(A[t, t^{-1}]) = K_1(A) \oplus N_+K_1(A) \oplus N_-K_1(A) \oplus K_0(A)$$

*and*

$$K_0(A[t, t^{-1}]) = K_0(A) \oplus N_+K_0(A) \oplus N_-K_0(A) \oplus K_{-1}(A) .$$

*Proof.* See [1], Theorem 7.4 of chapter XII.

We will also use the following well-known result.

**Proposition 4.** *If 2 is invertible in $A$ the groups $N_\pm K_1(A)$ are divisible by 2.*

*Proof.* Let $\alpha = 1 + \nu t$ represent an element of $N_+K_1(A)$ with $\nu$ a nilpotent matrix of size $n$. Let

$$P(X) = \sum_{k=0}^{\infty} \binom{1/2}{k} X^k \in \mathbb{Z}[1/2][X] .$$

Then $P(\nu t) \in M_n(A[t])$ and $(P(\nu t))^2 = 1 + \nu t$. This shows that $N_+K_1(A)$ is divisible by 2. To show uniqueness it suffices to show that $N_+K_1(A)$ has no 2-torsion. Take $\alpha = 1 + \nu t$ as before and suppose that $\alpha^2 \in E(A[t])$. Define $s = t(2 + \nu t)$, so that $\alpha^2 = 1 + \nu s$. Since

$$t = \sum_{k=1}^{\infty} \binom{1/2}{k} \nu^{k-1} s^k$$

we have $M_n(A)[t] = M_n(A)[s]$. If $\alpha^2 = 1 + \nu s \in E(A[s]) = E(M_n(A)[s])$ we clearly also have $\alpha = 1 + \nu t \in E(M_n(A)[t])$.

**Corollary 5.** *If 2 is invertible in $A$, then the groups $N_\pm K_0(A)$ are uniquely divisible by 2.*

*Proof.* $K_0(A)$ is a direct factor of $K_1(A[X, X^{-1}])$, hence $N_\pm K_0(A)$ is a direct factor of $N_\pm K_1(A[X, X^{-1}])$.
Assume now that $A$ has an involution. Associating to any projective module its dual and to any matrix its conjugate transpose yields actions of $\mathbb{Z}/2$ on $K_0$ and $K_1$ which are compatible with the decompositions of Theorem 3.

**Corollary 6.** *Suppose that $A$ is a ring with involution in which 2 is invertible. Then*

$$H^2(\mathbb{Z}/2, K_0(A[t])/K_0(A)) = 0 \ .$$

## 3. Polynomial rings

**Theorem 7.** *Let $A$ be an associative ring with involution, in which 2 is invertible. Let $\epsilon$ be 1 or $-1$ and let $W$ be the Witt group functor of $\epsilon$-hermitian spaces. The natural homomorphism*

$$W(A) \longrightarrow W(A[t])$$

*is an isomorphism.*

*Proof.* It suffices to show that the homomorphism $W(A[t]) \to W(A)$ given by the evaluation at $t = 0$ is an isomorphism. Surjectivity is obvious. To prove injectivity let $(P, \alpha)$ be a space over $A[t]$ and $(P(0), \alpha(0))$ its reduction modulo $t$. Suppose that $(P(0), \alpha(0))$ is isometric to some hyperbolic space $H(Q)$. Choosing a projective module $Q'$ such that $Q \oplus Q'$ is free and adding to $(P, \alpha)$ the space $H(Q'[t])$ we may assume that $P(0)$ is the hyperbolic space over a free module. The class of $P$ in $K_0(A[t])/K_0(A) = N_+(A)$ is a symmetric element. By Corollary 5 it can be written as $a + a^*$. Hence, adding to $(P, \alpha)$ a suitable free hyperbolic space, we may assume that $(P, \alpha)$ is of the form

$$H(A^n[t]) \perp (R \oplus R^*, \beta) \ .$$

Let $R'$ be an $A[t]$-module such that $R \oplus R'$ is free. Adding to $(P, \alpha)$ the hyperbolic space $H(R)$ we are reduced to the case in which $P$ is free and $\alpha$ is an invertible $\epsilon$-hermitian matrix with entries in $A[t]$.

**Lemma 8.** *Let $\alpha = \alpha^* \in \mathrm{M}_n(A[t])$ be any $\epsilon$-hermitian matrix. There exist an integer $m$ and a matrix $\tau \in \mathrm{GL}_{n+2m}(A[t])$ (actually in $\mathrm{E}_{n+2m}(A[t])$) such that*

$$\tau^* \begin{pmatrix} \alpha & 0 \\ 0 & \chi \end{pmatrix} \tau = \alpha_0 + t\alpha_1 \ ,$$

*where $\alpha_0$ and $\alpha_1$ are constant matrices and $\chi$ is a sum of hyperbolic blocs $\left(\begin{smallmatrix} 0 & 1 \\ \epsilon 1 & 0 \end{smallmatrix}\right)$ of various sizes.*

*Proof of the lemma.* Write $\alpha = \gamma + \delta t^N$, where $\delta$ is constant and $\gamma$ of degree less than $N$. Assume that $N$ is at least 2. Since $\delta$ is $\epsilon$-hermitian and 2 is invertible in $A$ we can write $\delta = \sigma + \epsilon\sigma^*$. Then

$$\begin{pmatrix} 1 & t & -\sigma^* t^{N-1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma + \sigma t^N + \epsilon\sigma^* t^N & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \epsilon & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ -\sigma t^{N-1} & 0 & 1 \end{pmatrix}$$

is of degree $\leq N - 1$ and after $N - 1$ such transformations we get a linear matrix.

Writing $\alpha_0 + t\alpha_1$ as $\alpha_0(1 + \nu t)$ we see immediately that, $\alpha$ being invertible, $\nu$ is nilpotent. The formal power series

$$\tau = (1 + \nu t)^{-1/2} = \sum \binom{-1/2}{k} (\nu t)^k$$

is a polynomial. From $\alpha = \epsilon\alpha^*$ we get $\nu^*\alpha_0^* = \alpha_0\nu$. This implies that $\tau^*\alpha_0^* = \alpha_0\tau$ and therefore

$$\tau^*\alpha\tau = \tau^*\alpha_0(1 + \nu t)\tau = \alpha_0\tau(1 + \nu t)\tau = \alpha_0 \ .$$

This proves that $(P, \alpha)$ is Witt equivalent to $(P(0), \alpha(0))$ and is, therefore, hyperbolic.

## References

1. H. Bass, *Algebraic K-Theory*, Benjamin.
2. M. Karoubi, *Localisation de formes quadratiques.II*, Ann. Sci. Éc. Norm. Sup. **8** (1975), 99–155.